

L_p -discrepancy of the symmetrized van der Corput sequence

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Abstract

It is well known that the L_p -discrepancy for $p \in [1, \infty]$ of the van der Corput sequence is of exact order of magnitude $O((\log N)/N)$. This however is for $p \in (1, \infty)$ not best possible with respect to the lower bounds according to Roth and Proinov. For the case $p = 2$ it is well known that the symmetrization trick due to Davenport leads to the optimal L_2 -discrepancy rate $O(\sqrt{\log N}/N)$ for the symmetrized van der Corput sequence. In this note we show that this result holds for all $p \in (1, \infty)$. The proof is based on an estimate of the Haar coefficients of the corresponding local discrepancy and on the use of the Littlewood-Paley inequality.

Keywords: L_p -discrepancy, van der Corput sequence, Davenport's reflection principle

MSC 2000: 11K38, 11K31

1 Introduction and Statement of the Result

For an infinite sequence $\mathcal{S} = (x_n)_{n \geq 0}$ of points in $[0, 1)$ the *local discrepancy* of its first N elements is defined as

$$D_N(\mathcal{S}, t) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[0, t)}(x_n) - t,$$

where, throughout this paper, $\mathbf{1}_I(x)$ denotes the indicator function of the interval $I \subseteq [0, 1]$. The L_p -discrepancy for $p \in [1, \infty]$ of \mathcal{S} is defined as the L_p -norm of the local discrepancy, thus, for $p \in (1, \infty)$

$$L_{p,N}(\mathcal{S}) := \|D_N(\mathcal{S}, \cdot)\|_{L_p} = \left(\int_0^1 |D_N(\mathcal{S}, t)|^p dt \right)^{1/p}.$$

For $p = \infty$ we have

$$L_{\infty,N}(\mathcal{S}) := \|D_N(\mathcal{S}, \cdot)\|_{L_\infty} = \sup_{t \in [0, 1]} |D_N(\mathcal{S}, t)|.$$

The L_p -discrepancy is a quantitative measure for the irregularity of distribution of a sequence modulo one, see, e.g., [6, 15]. It is also related to the worst-case integration error of a quasi-Monte Carlo rule, see, e.g., [5, 14].

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It is well known that for every $p \in (1, \infty]$ there exists a positive number c_p with the property that for every sequence \mathcal{S} in $[0, 1)$ we have

$$L_{\infty, N}(\mathcal{S}) \geq c_{\infty} \frac{\log N}{N} \quad \text{for infinitely many } N \in \mathbb{N} \quad (1)$$

and, for $p \in (1, \infty)$,

$$L_{p, N}(\mathcal{S}) \geq c_p \frac{\sqrt{\log N}}{N} \quad \text{for infinitely many } N \in \mathbb{N}, \quad (2)$$

where \log denotes the natural logarithm and where \mathbb{N} denotes the set of positive integers $\{1, 2, 3, \dots\}$. The result for $p = \infty$ was shown by Schmidt [24] and the result for $p \in (1, \infty)$ was shown by Proinov [20] based on results of Roth [23] and Schmidt [25]. Both lower bounds (1) and (2) are optimal in the order of magnitude in N .

A prototype of a sequence with low discrepancy is the *van der Corput sequence* (in base 2). Let $\varphi(n)$ denote the *radical inverse* of $n \in \mathbb{N}_0$ in base 2 (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) which is defined as $\varphi(n) := \sum_{i=0}^k n_i 2^{-i-1}$ whenever n has binary expansion $n = \sum_{i=0}^k n_i 2^i$, where $n_i \in \{0, 1\}$ for all $i \in \{0, \dots, k\}$. The van der Corput sequence (in base 2) is the sequence $\mathcal{V} = (y_n)_{n \geq 0}$ where $y_n = \varphi(n)$ for $n \in \mathbb{N}_0$.

For the van der Corput sequence it is known that (see, e.g. [1, 9])

$$\limsup_{N \rightarrow \infty} \frac{NL_{\infty, N}(\mathcal{V})}{\log N} = \frac{1}{3 \log 2}$$

and hence $L_{\infty, N}(\mathcal{V})$ is of order of magnitude $O((\log N)/N)$ which is best possible in N according to (1). However, for $p \in (1, \infty)$ it is known that (see, e.g., [3, 22] for $p = 2$ and [19] for general p)

$$\limsup_{N \rightarrow \infty} \frac{NL_{p, N}(\mathcal{V})}{\log N} = \frac{1}{6 \log 2}.$$

This means that $L_{p, N}(\mathcal{V})$ for $p \in (1, \infty)$ is only of order of magnitude $O((\log N)/N)$ which is *not* best possible in N if we compare with (2).

One way out to overcome this defect of the van der Corput sequence is based on symmetrization which was initially introduced by Davenport for $(n\alpha)$ -sequences (see [6, Theorem 1.75]). This method is also known as *Davenport's reflection principle*.

We define the *symmetrized van der Corput sequence* (in base 2) $\mathcal{V}^{\text{sym}} = (z_n)_{n \geq 0}$ as

$$z_n = \begin{cases} \varphi(m) & \text{if } n = 2m, \\ 1 - \varphi(m) & \text{if } n = 2m + 1. \end{cases}$$

Then it is known, see e.g. [3, 8, 12, 21], that the L_2 -discrepancy of the symmetrized van der Corput sequence is of optimal order of magnitude in N compared to the lower bound in (2), i.e.,

$$L_{2, N}(\mathcal{V}^{\text{sym}}) \ll \frac{\sqrt{\log N}}{N}. \quad (3)$$

Here and throughout the paper, for functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$, we write $g(N) \ll f(N)$, if there exists a $C > 0$ such that $g(N) \leq Cf(N)$ for all $N \in \mathbb{N}$, $N \geq 2$. If we would like to stress that C depends on some parameter, say p , this will be indicated by writing \ll_p .

It is the aim of this paper to show that the estimate (3) holds for all $p \in (1, \infty)$. We show:

Theorem 1 For every $p \in (1, \infty)$ we have

$$L_{p,N}(\mathcal{V}^{\text{sym}}) \ll_p \frac{\sqrt{\log N}}{N}.$$

The proof of this result is based on the Haar function system (in base 2) and will be given in Section 3. First we collect some auxiliary results in the following section.

2 Auxiliary Results

In order to estimate the L_p -discrepancy of \mathcal{V}^{sym} we use the one-dimensional Haar system. Haar functions are a useful and often applied tool in discrepancy theory, see e.g. [4, 10, 16, 17, 18].

To begin with, a *dyadic interval* of length 2^{-j} , $j \in \mathbb{N}_0$, in $[0, 1)$ is an interval of the form

$$I = I_{j,m} := \left[\frac{m}{2^j}, \frac{m+1}{2^j} \right) \quad \text{for } m = 0, 1, \dots, 2^j - 1.$$

We also define $I_{-1,0} = [0, 1)$. The left and right half of $I = I_{j,m}$ are the dyadic intervals $I^+ = I_{j,m}^+ = I_{j+1,2m}$ and $I^- = I_{j,m}^- = I_{j+1,2m+1}$, respectively. The *Haar function* $h_I = h_{j,m}$ with support I is the function on $[0, 1)$ which is +1 on the left half of I , -1 on the right half of I and 0 outside of I . The L_∞ -normalized *Haar system* consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_0$ and $m = 0, 1, \dots, 2^j - 1$ together with the indicator function $h_{-1,0}$ of $[0, 1)$. Normalized in $L_2([0, 1))$ we obtain the *orthonormal Haar basis* of $L_2([0, 1))$.

The *Haar coefficients* of a function $f \in L_p([0, 1))$ are defined as

$$\mu_{j,m}(f) := \langle f, h_{j,m} \rangle = \int_0^1 f(t) h_{j,m}(t) dt \quad \text{for } j \in \mathbb{N}_{-1} \text{ and } m \in \mathbb{D}_j,$$

where here and later on we use the abbreviations $\mathbb{N}_{-1} := \mathbb{N}_0 \cup \{-1\}$, $\mathbb{D}_j := \{0, 1, \dots, 2^j - 1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} := \{0\}$.

In the following, we will compute the *Haar coefficients* of the local discrepancy of \mathcal{V}^{sym} , i.e.,

$$\mu_{j,m}(D_N(\mathcal{V}^{\text{sym}}, \cdot)) = \langle D_N(\mathcal{V}^{\text{sym}}, \cdot), h_{j,m} \rangle = \int_0^1 D_N(\mathcal{V}^{\text{sym}}, t) h_{j,m}(t) dt.$$

Preceding the computation of the Haar coefficients, we collect some properties of the radical inverse function $\varphi(n)$ which we will need in the proof of the essential Lemma 5.

Lemma 1 The following relations hold for the radical inverse function φ :

1. $\varphi(2^j s) = \frac{1}{2^j} \varphi(s)$ for all $j, s \in \mathbb{N}_0$,
2. $\varphi(2^j \varphi(m)) = \frac{m}{2^j}$ for all $j \in \mathbb{N}_0$ and $m \in \{0, \dots, 2^j - 1\}$,
3. $\varphi(n) \in I_{j,m}$ if and only if $n = 2^j \varphi(m) + 2^j s$ for some $s \in \mathbb{N}_0$,
4. $\frac{A}{2} \leq \sum_{s=0}^A |1 - 2\varphi(s)| \leq \frac{A}{2} + 1$ for all $A \in \mathbb{N}_0$.

Proof. 1. Let $s = \sum_{i=0}^k s_i 2^i$, where $s_i \in \{0, 1\}$ for all $i \in \{0, \dots, k\}$. Then $\varphi(2^j s) = \varphi(\sum_{i=0}^k s_i 2^{i+j}) = \sum_{i=0}^k s_i 2^{-i-j-1} = 2^{-j} \sum_{i=0}^k s_i 2^{-i-1} = \frac{1}{2^j} \varphi(s)$.

2. Since $0 \leq m \leq 2^j - 1$, m has a binary representation of the form $m = \sum_{i=0}^{j-1} m_i 2^i$, where $m_i \in \{0, 1\}$ for all $i \in \{0, \dots, j-1\}$. Then $2^j \varphi(m) = \sum_{i=0}^{j-1} m_i 2^{j-i-1}$ and therefore $\varphi(2^j \varphi(m)) = \sum_{i=0}^{j-1} m_i 2^{-(j-i-1)-1} = 2^{-j} \sum_{i=0}^{j-1} m_i 2^i = m 2^{-j}$.
3. We write n in the form $n = \tilde{n} + 2^j s$, where $\tilde{n} \in \{0, 1, \dots, 2^j - 1\}$ and $s \in \mathbb{N}_0$. Then $\varphi(\tilde{n}) = n' 2^{-j}$ for some $n' \in \{0, 1, \dots, 2^j - 1\}$ as one can verify easily. We have $\varphi(n) = \frac{n'}{2^j} + \frac{\varphi(s)}{2^j}$. We see that $\varphi(n) \in I_{j,m}$ is true if and only if $n' = m$. But from Point 2. we know $\varphi^{-1}(m 2^{-j}) = 2^j \varphi(m)$ and the proof of Point 3. is done.
4. To begin with, we verify the relation $|1 - 2\varphi(2n)| + |1 - 2\varphi(2n + 1)| = 1$ for all $n \in \mathbb{N}_0$. Therefore we observe that $\varphi(2n) \leq \frac{1}{2}$ for all $n \in \mathbb{N}_0$. We also have $\varphi(2n + 1) = \varphi(2n) + \varphi(1) = \varphi(2n) + \frac{1}{2}$, hence

$$\begin{aligned} |1 - 2\varphi(2n)| + |1 - 2\varphi(2n + 1)| &= 1 - 2\varphi(2n) + \left| 1 - 2\left(\varphi(2n) + \frac{1}{2}\right) \right| \\ &= 1 - 2\varphi(2n) + | - 2\varphi(2n) | = 1. \end{aligned}$$

This leads to

$$\sum_{s=0}^A |1 - 2\varphi(s)| = \sum_{n=0}^{(A-1)/2} \{|1 - 2\varphi(2n)| + |1 - 2\varphi(2n + 1)|\} = \sum_{n=0}^{(A-1)/2} 1 = \frac{A+1}{2}$$

for odd A and

$$\begin{aligned} \sum_{s=0}^A |1 - 2\varphi(s)| &= \sum_{n=0}^{(A-2)/2} \{|1 - 2\varphi(2n)| + |1 - 2\varphi(2n + 1)|\} + |1 - 2\varphi(A)| \\ &= \sum_{n=0}^{(A-2)/2} 1 + |1 - 2\varphi(A)| = \frac{A}{2} + |1 - 2\varphi(A)| \end{aligned}$$

for even A .

Hence in both cases we have $\frac{A}{2} \leq \sum_{s=0}^A |1 - 2\varphi(s)| \leq \frac{A}{2} + 1$.

□

Lemma 2 *The Haar coefficient $\mu_{-1,0}$ of the local discrepancy $D_N(\mathcal{V}^{\text{sym}}, \cdot)$ satisfies*

$$|\mu_{-1,0}| = \begin{cases} 0 & \text{if } N = 2M, \\ \left| \frac{1}{2N} - \frac{\varphi(M)}{N} \right| \leq \frac{1}{2N} & \text{if } N = 2M + 1. \end{cases}$$

Proof. We have

$$\begin{aligned} \mu_{-1,0} &= \int_0^1 D_N(\mathcal{V}^{\text{sym}}, t) dt = \int_0^1 \left(\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[0,t)}(x_n) - t \right) dt \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \int_0^1 \mathbf{1}_{[0,t)}(x_n) dt - \frac{1}{2} = \frac{1}{N} \sum_{n=0}^{N-1} \left(\int_{x_n}^1 \mathbf{1}_{[0,t)}(x_n) dt \right) - \frac{1}{2} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} (1 - x_n) - \frac{1}{2} = \frac{1}{2} - \frac{1}{N} \sum_{n=0}^{N-1} x_n. \end{aligned}$$

We therefore have to investigate the sum $\sum_{n=0}^{N-1} x_n$. If $N = 2M$, then we have

$$\sum_{n=0}^{2M-1} x_n = \sum_{n=0}^{M-1} \varphi(m) + \sum_{m=0}^{M-1} (1 - \varphi(m)) = M = \frac{N}{2}.$$

For $N = 2M + 1$, we find

$$\sum_{n=0}^{2M} x_n = \sum_{n=0}^{2M-1} x_n + \varphi(M) = M + \varphi(M) = \frac{N-1}{2} + \varphi(M).$$

This leads to the desired result. \square

Remark 1 Lemma 2 is a crucial fact. It shows how the symmetrization trick keeps the Haar coefficient $\mu_{-1,0}$ small enough in order to achieve the optimal L_p -discrepancy rate. Let us compare this with the behavior of the Haar coefficient $\mu_{-1,0}^{N,\varphi}$ of the local discrepancy of the first N terms of the *usual (not symmetrized)* van der Corput sequence: for $2^m \leq N < 2^{m+1}$ it is known that

$$\mu_{-1,0}^{N,\varphi} = \frac{1}{2N} \left(1 + \sum_{r=0}^{m-1} \left\| \frac{N}{2^{r+1}} \right\| \right),$$

where $\|x\|$ denotes the distance of x to the nearest integer, see, e.g., [7, Proposition 1]. Further we know from [13, Theorem 3] that

$$\max_{2^m \leq N < 2^{m+1}} \sum_{r=0}^{m-1} \left\| \frac{N}{2^{r+1}} \right\| = \frac{m}{3} + \frac{1}{9} - (-1)^m \frac{1}{9 \cdot 2^m}.$$

Hence it follows that there exists some constant $c > 0$ such that

$$\mu_{-1,0}^{N,\varphi} \geq c \frac{\log N}{N} \quad \text{for infinitely many } N \in \mathbb{N}.$$

Therefore, for the usual van der Corput sequence, already the first (and only the first, c.f. Lemma 5) Haar coefficient of the local discrepancy has the “bad” order of magnitude $(\log N)/N$.

In the following, we write $D_N^{\text{sym}}(t) := D_N(\mathcal{V}^{\text{sym}}, t)$ and denote by $D_N^\varphi(t)$ the local discrepancy of the first N elements of the sequence $(\varphi(n))_{n \geq 0}$ and analogously $D_N^{1-\varphi}(t)$ the local discrepancy of the first N elements of the sequence $(1 - \varphi(n))_{n \geq 0}$. Let $\mu_{j,m}^{N,\text{sym}}$, $\mu_{j,m}^{N,\varphi}$ and $\mu_{j,m}^{N,1-\varphi}$ be the Haar coefficients of these three functions.

Lemma 3 For $N = 2M$ we have

$$D_{2M}^{\text{sym}}(t) = \frac{1}{2} \left(D_M^\varphi(t) + D_M^{1-\varphi}(t) \right)$$

and for $N = 2M + 1$

$$D_{2M+1}^{\text{sym}}(t) = \frac{1}{2M+1} \left((M+1) D_{M+1}^\varphi(t) + M D_M^{1-\varphi}(t) \right).$$

Proof. For $N = 2M$ we have

$$\begin{aligned} D_{2M}^{\text{sym}}(t) &= \frac{1}{2M} \left(\sum_{n=0}^{M-1} \mathbf{1}_{[0,t)}(\varphi(n)) + \sum_{n=0}^{M-1} \mathbf{1}_{[0,t)}(1 - \varphi(n)) \right) - t \\ &= \frac{1}{2} \left(\frac{1}{M} \sum_{n=0}^{M-1} \mathbf{1}_{[0,t)}(\varphi(n)) - t + \frac{1}{M} \sum_{n=0}^{M-1} \mathbf{1}_{[0,t)}(1 - \varphi(n)) - t \right) \\ &= \frac{1}{2} \left(D_M^\varphi(t) + D_M^{1-\varphi}(t) \right). \end{aligned}$$

and for $N = 2M + 1$ we obtain

$$\begin{aligned} D_{2M+1}^{\text{sym}}(t) &= \frac{1}{2M+1} \left(\sum_{n=0}^M \mathbf{1}_{[0,t)}(\varphi(n)) + \sum_{n=0}^{M-1} \mathbf{1}_{[0,t)}(1 - \varphi(n)) \right) - t \\ &= \frac{1}{2M+1} \left(\sum_{n=0}^M \mathbf{1}_{[0,t)}(\varphi(n)) - (M+1)t + \sum_{n=0}^{M-1} \mathbf{1}_{[0,t)}(1 - \varphi(n)) - Mt \right) \\ &= \frac{1}{2M+1} \left((M+1)D_{M+1}^\varphi(t) + MD_M^{1-\varphi}(t) \right). \end{aligned}$$

□

Corollary 1 *We have*

$$|\mu_{j,m}^{N,\text{sym}}| \leq \begin{cases} \frac{1}{2} \left(|\mu_{j,m}^{M,\varphi}| + |\mu_{j,m}^{M,1-\varphi}| \right) & \text{if } N = 2M, \\ \frac{1}{2M+1} \left((M+1)|\mu_{j,m}^{M+1,\varphi}| + M|\mu_{j,m}^{M,1-\varphi}| \right) & \text{if } N = 2M+1. \end{cases}$$

Proof. We consider linearity of integration and the triangle inequality to obtain the result from Lemma 3. □

We proceed with the calculation of $\mu_{j,m}$ in the case $j \in \mathbb{N}_0$ and first prove the following general lemma.

Lemma 4 *Let $j \in \mathbb{N}_0$ and $m \in \mathbb{D}_j$. Then for the volume part $f(t) = t$ of the discrepancy function we have*

$$\mu_{j,m}(f) = -2^{-2j-2}$$

and for the counting part $g(t) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[0,t)}(x_n)$ we have

$$\mu_{j,m}(g) = \frac{2^{-j-1}}{N} \sum_{\substack{n=0 \\ x_n \in \overset{\circ}{I}_{j,m}}}^{N-1} (|2m+1 - 2^{j+1}x_n| - 1),$$

where $\overset{\circ}{I}_{j,m} = \left(\frac{m}{2^j}, \frac{m+1}{2^j} \right)$ denotes the interior of $I_{j,m}$.

Proof. Of course,

$$\mu_{j,m}(f) = \int_0^1 t h_{j,m}(t) dt = \int_{m2^{-j}}^{m2^{-j}+2^{-j-1}} t dt - \int_{m2^{-j}+2^{-j-1}}^{(m+1)2^{-j}} t dt = -2^{-2j-2}.$$

The Haar coefficients of g are given by

$$\mu_{j,m}(g) = \int_0^1 \left(\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[0,t)}(x_n) h_{j,m}(t) \right) dt = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{\int_0^1 \mathbf{1}_{[0,t)}(x_n) h_{j,m}(t) dt}_{\mathcal{I}_n}.$$

We analyse \mathcal{I}_n . If $x_n \notin I_{j,m}$ or $x_n = \frac{m}{2^j}$, it is evident that $\mathcal{I}_n = 0$. One can check by simple integration, that in the case $x_n \in I_{j,m}^+$ we have $\mathcal{I}_n = m2^{-j} - x_n$, and if $x_n \in I_{j,m}^-$, then $\mathcal{I}_n = -2^{-j} - (m2^{-j} - x_n)$. These results can be combined to

$$\mathcal{I}_n = 2^{-j-1}(|2m + 1 - 2^{j+1}x_n| - 1) \quad \text{if } x_n \in \overset{\circ}{I}_{j,m}.$$

The claimed result follows. \square

Now we are ready to show a central lemma.

Lemma 5 *We have*

$$|\mu_{j,m}^{N,\varphi}| \leq \frac{1}{N} \frac{1}{2^j} \quad \text{and} \quad |\mu_{j,m}^{N,1-\varphi}| \leq \frac{1}{N} \frac{1}{2^j}$$

for all $0 \leq j < \lceil \log_2 N \rceil$ and

$$|\mu_{j,m}^{N,\varphi}| = |\mu_{j,m}^{N,1-\varphi}| = 2^{-2j-2}$$

for all $j \geq \lceil \log_2 N \rceil$.

Proof. We start with $x_n = \varphi(n)$ and investigate the sum

$$\sum_{\substack{n=0 \\ \varphi(n) \in \overset{\circ}{I}_{j,m}}}^{N-1} (|2m + 1 - 2^{j+1}\varphi(n)| - 1),$$

which, according to Lemma 1, we can transfer to

$$\begin{aligned} & \sum_{s=1}^A \left(|2m + 1 - 2^{j+1}\varphi(2^j\varphi(m) + 2^j s)| - 1 \right) \\ &= \sum_{s=1}^A \left(\left| 2m + 1 - 2^{j+1} \left(\frac{m}{2^j} + \varphi(2^j s) \right) \right| - 1 \right) \\ &= \sum_{s=1}^A \left(|1 - 2^{j+1}\varphi(2^j s)| - 1 \right) = \sum_{s=0}^A (|1 - 2\varphi(s)| - 1). \end{aligned}$$

We still have to specify the upper index. The conditions $0 \leq n \leq N-1$ and $n = 2^j\varphi(m) + 2^j s$ lead to $s \leq \frac{N-1}{2^j} - \varphi(m)$, this is why we choose $A := \lfloor \frac{N-1}{2^j} - \varphi(m) \rfloor$. We also see that there are no elements of $\{x_0, x_1, \dots, x_{N-1}\}$ contained in $\overset{\circ}{I}_{j,m}$, if $2^j\varphi(m) + 2^j \geq N$, which is fulfilled if $2^j \geq N$ or $j \geq \lceil \log_2 N \rceil$. Regarding that fact, we immediately conclude from Lemma 4 that $|\mu_{j,m}^{N,\varphi}| = 2^{-2j-2}$ for $j \geq \lceil \log_2 N \rceil$. We proceed with the case $j < \lceil \log_2 N \rceil$ and have

$$\begin{aligned} \mu_{j,m}^{N,\varphi} &= \frac{2^{-j-1}}{N} \sum_{s=0}^A (|1 - 2\varphi(s)| - 1) + 2^{-2j-2} \\ &\leq \frac{2^{-j-1}}{N} \left(\frac{A}{2} + 1 - (A+1) \right) + 2^{-2j-2} \\ &= -\frac{2^{-j-2}}{N} A + 2^{-2j-2} \leq -\frac{2^{-j-2}}{N} \left(\frac{N-1}{2^j} - \varphi(m) - 1 \right) + 2^{-2j-2} \end{aligned}$$

$$= \frac{1}{N} \left(2^{-2j-2} + (\varphi(m) + 1) 2^{-j-2} \right) \leq \frac{1}{N} \left(2^{-2j-2} + 2^{-j-1} \right).$$

We also find

$$\begin{aligned} \mu_{j,m}^{N,\varphi} &\geq \frac{2^{-j-1}}{N} \left(\frac{A}{2} - (A+1) \right) + 2^{-2j-2} \\ &= -\frac{2^{-j-1}}{N} \left(\frac{A}{2} + 1 \right) + 2^{-2j-2} \geq -\frac{2^{-j-1}}{N} \left(\frac{N-1}{2^{j+1}} - \frac{\varphi(m)}{2} + 1 \right) + 2^{-2j-2} \\ &= \frac{1}{N} \left(2^{-2j-2} + \left(\frac{\varphi(m)}{2} - 1 \right) 2^{-j-1} \right) \geq \frac{1}{N} \left(2^{-2j-2} - 2^{-j-1} \right). \end{aligned}$$

By combining these results we finally obtain

$$|\mu_{j,m}^{N,\varphi}| \leq \frac{1}{N} \left(2^{-2j-2} + 2^{-j-1} \right) \leq \frac{1}{N} \frac{1}{2^j}$$

as claimed.

We turn to the estimation of $|\mu_{j,m}^{N,1-\varphi}|$, which can be treated similarly to $|\mu_{j,m}^{N,\varphi}|$. To begin with, we observe that

$$\begin{aligned} &\sum_{\substack{n=0 \\ 1-\varphi(n) \in I_{j,m}^{\circ}}}^{N-1} \left(|2m+1 - 2^{j+1}(1-\varphi(n))| - 1 \right) \\ &= \sum_{\substack{n=0 \\ \varphi(n) \in I_{j,2j-m-1}^{\circ}}}^{N-1} \left(|2m+1 - 2^{j+1}(1-\varphi(n))| - 1 \right) \\ &= \sum_{s=1}^B \left(\left| 2m+1 - 2^{j+1} \left(1 - \varphi \left(2^j \varphi(2^j - m - 1) + 2^j s \right) \right) \right| - 1 \right) \\ &= \sum_{s=1}^B \left(\left| 2m+1 - 2^{j+1} \left(1 - \left(\frac{2^j - m - 1}{2^j} + \varphi(2^j s) \right) \right) \right| - 1 \right) \\ &= \sum_{s=1}^B (|-1 + 2\varphi(s)| - 1) = \sum_{s=0}^B (|1 - 2\varphi(s)| - 1). \end{aligned}$$

In this expression, $B := \lfloor \frac{N-1}{2^j} - \varphi(2^j - m - 1) \rfloor$, which we deduce in the same way as the upper index A above. Completely analogously as above, we obtain

$$|\mu_{j,m}^{N,1-\varphi}| \leq \frac{1}{N} \left(2^{-2j-2} + 2^{-j-1} \right) \leq \frac{1}{N} \frac{1}{2^j}$$

for $j < \lceil \log_2 N \rceil$. The case $j \geq \lceil \log_2 N \rceil$ also follows the same lines as above. \square

Corollary 2 *The Haar coefficients of the symmetrized van der Corput sequence for $j \in \mathbb{N}_0$ fulfil*

$$|\mu_{j,m}^{N,\text{sym}}| \begin{cases} \leq \frac{1}{2^{j-1}} \frac{1}{N} & \text{if } j < \lceil \log_2 N \rceil, \\ = 2^{-2j-2} & \text{if } j \geq \lceil \log_2 N \rceil. \end{cases}$$

Proof. We combine Corollary 1 and Lemma 5 to obtain the result. \square

3 The Proof of Theorem 1

We are ready to show that the L_p -discrepancy of the symmetrized van der Corput sequence has optimal order in N for any $p \in (1, \infty)$. We apply the *Littlewood-Paley inequality* which involves the *square function*

$$S(f) := \left(\sum_{j \in \mathbb{N}_{-1}} \sum_{m \in \mathbb{D}_j} 2^{2 \max\{0, j\}} \langle f, h_{j,m} \rangle^2 \mathbf{1}_{I_{j,m}} \right)^{1/2}$$

of a function $f \in L_p([0, 1])$.

Lemma 6 (Littlewood-Paley inequality) *Let $p \in (1, \infty)$. Then there exist $c_p, C_p > 0$ such that for every $f \in L_p([0, 1])$ we have*

$$c_p \|f\|_{L_p} \leq \|S(f)\|_{L_p} \leq C_p \|f\|_{L_p}.$$

Proofs of these inequalities and further details also yielding the right asymptotic behavior of the involved constants can be found in [2, 28, 29]. Littlewood-Paley theory has already been used in the context of discrepancy before, see, e.g., [4, 11, 26, 27].

Now we can give the proof of Theorem 1.

Proof. Throughout this proof, we simply write $\mu_{j,m}$ instead of $\mu_{j,m}^{N, \text{sym}}$. Using Lemma 6 with $f = D_N(\mathcal{V}^{\text{sym}}, \cdot)$ we have

$$\begin{aligned} L_{p,N}(\mathcal{V}^{\text{sym}}) &= \|D_N(\mathcal{V}^{\text{sym}}, \cdot)\|_{L_p} \\ &\ll_p \|S(D_N(\mathcal{V}^{\text{sym}}, \cdot))\|_{L_p} \\ &= \left\| \left(\sum_{j \in \mathbb{N}_{-1}} \sum_{m \in \mathbb{D}_j} 2^{2 \max\{0, j\}} \mu_{j,m}^2 \mathbf{1}_{I_{j,m}} \right)^{1/2} \right\|_{L_p} \\ &= \left\| \sum_{j \in \mathbb{N}_{-1}} 2^{2 \max\{0, j\}} \sum_{m \in \mathbb{D}_j} \mu_{j,m}^2 \mathbf{1}_{I_{j,m}} \right\|_{L_{p/2}}^{1/2} \\ &\leq \left(\sum_{j \in \mathbb{N}_{-1}} 2^{2 \max\{0, j\}} \left\| \sum_{m \in \mathbb{D}_j} \mu_{j,m}^2 \mathbf{1}_{I_{j,m}} \right\|_{L_{p/2}} \right)^{1/2}, \end{aligned}$$

where we used Minkowski's inequality for the $L_{p/2}$ -norm. Hence, in order to prove the result it suffices to show that

$$\sum_{j \in \mathbb{N}_{-1}} 2^{2 \max\{0, j\}} \left\| \sum_{m \in \mathbb{D}_j} \mu_{j,m}^2 \mathbf{1}_{I_{j,m}} \right\|_{L_{p/2}} \ll \frac{\log N}{N^2}.$$

Now Lemma 2 and Lemma 5 give

$$\sum_{j \in \mathbb{N}_{-1}} 2^{2 \max\{0, j\}} \left\| \sum_{m \in \mathbb{D}_j} \mu_{j,m}^2 \mathbf{1}_{I_{j,m}} \right\|_{L_{p/2}} \leq \frac{1}{4N^2} \|\mathbf{1}_{[0,1]}\|_{L_{p/2}} + \frac{4}{N^2} \sum_{j=0}^{\lceil \log_2 N \rceil - 1} \left\| \sum_{m \in \mathbb{D}_j} \mathbf{1}_{I_{j,m}} \right\|_{L_{p/2}}$$

$$\begin{aligned}
& + \frac{1}{16} \sum_{j=\lceil \log_2 N \rceil}^{\infty} 2^{-2j} \left\| \sum_{m \in \mathbb{D}_j} \mathbf{1}_{I_{j,m}} \right\|_{L_{p/2}} \\
& \leq \frac{1}{4N^2} + \frac{4}{N^2} (\log_2 N + 1) + \frac{1}{12} \frac{1}{4^{\lceil \log_2 N \rceil}} \ll \frac{\log N}{N^2}
\end{aligned}$$

where we regarded the fact that $\sum_{m \in \mathbb{D}_j} \mathbf{1}_{I_{j,m}} = 1$ for a fixed $j \in \mathbb{N}_0$. The proof is complete. \square

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